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Coverings by Rook Domains

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ABSTRACT

It is shown that a 1-dense set in V_n^k must contain at least $n^{k-1}/(k-1)$ points. As a corollary, a conjecture of Golomb and Posner on error-distributing codes is proved. It is also shown that a $(k-2)$-dense set must contain at least $n^2/(k-1)$ points. Equality can be attained if and only if $k-1$ divides n and there are $k-2$ orthogonal latin squares of order $n/(k-1)$.

I. INTRODUCTION

V_n^k is defined to be the set of all k-tuples $(j_1, \ldots, j_k)$ of positive integers $\leqslant n$. This becomes a metric space if we define the distance between two points to be the number of pairs of corresponding entries which are distinct. The *j-dimensional rook domain* $(0 \leqslant j \leqslant k)$ of a point in V_n^k is defined as the union of the $\binom{k}{j}$ j-dimensional coordinate planes through the point, i.e., the set of points within distance j of the given point. This terminology, suggested by the picture when $j = 1$, $k = 2$, $n = 8$, was introduced by Golomb and Posner [1].

A j-dense set in V_n^k is a set of points whose j-dimensional rook domains cover V_n^k; thus each point is within distance j of some point of the set. The question considered here is the following: How many points must a j-dense set in V_n^k contain? The results available suggest that, for $0 \leqslant j \leqslant k-1$, the lower bound $n^{k-j}/\binom{k-1}{j}$ is valid, and the best for infinitely many values of n. This bound is trivial for $j = 0$ and $j = k-1$. The proofs for $j = 1$ and $k-2$ are given below. The bound $n^3/6$ for

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$k = 5$, $j = 2$ was verified by the author, but it was not clear how the proof could be generalized.

The theorem proved in Section 2 was stated in a weaker form in [1]. It allows some of the lower bounds given there for $w(k, n)$ (assuming the theorem) to be strengthened.

II. One-Dimensional Coverings

Here we consider a covering of V_n^k by 1-dimensional rook domains. It is convenient to adopt the convention that a point is covered k times by its own rook domain, thus taking the basic sets of the covering to be the rows which make up the rook domains.

The following lemma will be used to restrict the cases in which the lower bound can be attained:

Lemma 1. *Let a covering of V_n^k by rook domains have the property that if any point is covered more than once, it is covered k times—once by a row in each direction. Then the k-covered points form a number of k-cubes at distance $\geqslant 3$ from each other. If the number of rook domains is $n^{k-1}/(k-1)$, there are $(k-1)^{k-2}$ of these cubes, each of side $n/(k-1)$.*

Proof: By a cube we mean a set formed from V_n^k by removing an equal number of hyperplanes in each direction.

If two k-covered points are at distance 2, form the coordinate rectangle with these points as opposite vertices. The sides of this rectangle are rows of the covering rook domains. Hence the other two vertices of the rectangle are also covered k times. It follows easily that, if we take the maximal set of k-covered points obtained from one point by repeated addition of points at distance $\leqslant 2$ from the set, this set is a k-dimensional rectangular array, after permutation of the hyperplanes of V_n^k. Any center of a rook domain which is not in the set is at distance $\geqslant 3$. Hence this set is covered by rook domains which are centered in the set. The number of rows in each direction intersecting the set must then be the same, so that the set is a k-cube.

Let S be the set of centers of the covering rook domains. Consider a cube of k-covered points, with m in each row. Since there are m^{k-1} rows through the cube in each direction, this cube contains m^{k-1} points of S. Each row of the cube can contain at most one point of S. Hence every row contains one point, and any j-dimensional plane in the cube, $1 \leqslant j \leqslant k$, contains m^{j-1} points of S.

Let the n^{k-2} planes in the $(1, 2)$ direction be numbered by the index r. The r-th plane P_r can intersect at most one of the cubes of k-covered points. Let m_r be the length of the side of this cube, if it exists, m_r is zero otherwise. Then there are m_r points of S in P_r, with rook domains covering $n^2 - (n - m_r)^2$ points of P_r. The other points of the cube of side m_r have rook domains covering no additional points of P_r. There are $(k-2)(m_r^2 - m_r)$ of these points at distance 1 from P_r. Every other point of S at distance 1 from P_r has one additional point of P_r in its rook domain, and these points are all distinct. Hence if ν_r is the number of points of S at distance 1 from P_r,

$$(n - m_r)^2 = \nu_r - (k-2)(m_r^2 - m_r),$$

$$\frac{1}{k-1} n^2 - 2nm_r + (k-1)\, m_r^2 = -\frac{k-2}{k-1} n^2 + \nu_r + (k-2)\, m_r .$$

Summing over r,

$$(k-1) \sum \left(\frac{n}{k-1} - m_r\right)^2 = -\frac{k-2}{k-1} n^k + \sum [\nu_r + (k-2)\, m_r].$$

The quantity $\nu_r + (k-2)\, m_r$ is the sum of the numbers of elements of S in the $k - 2$ 3-dimensional planes through P_r. Hence by symmetry,

$$\sum [\nu_r + (k-2)\, m_r] = n(k-2)\nu,$$

if ν is the number of points of S. In particular, if $\nu = n^{k-1}/(k-1)$,

$$\sum \left(\frac{n}{k-1} - m_r\right)^2 = 0,$$

so that each m_r is $n/(k-1)$. The number of cubes of k-covered points is

$$\nu \Big/ \left(\frac{n}{k-1}\right)^{k-1} = (k-1)^{k-2}.$$

This completes the proof of the lemma.

THEOREM 1. *For $k \geqslant 2$, a 1-dense set in V_n^k has at least $n^{k-1}/(k-1)$ elements. This lower bound cannot be attained unless n is divisible by $k - 1$.*

PROOF: For $k = 2$, the conclusion is clear, for any set of m rook domains $(m < n)$ leave uncovered at least a subarry of $(n - m) \times (n - m)$ points. In the following it will be assumed that $k \geqslant 3$.

Let S be a 1-dense set in V_n^k. Consider the covering of V_n^k by the 1-dimensional rook domains of the points in S. For any point P of V_n^k, if c_j rows in the j-th direction cover P, the total number of times P is covered is

$$m(P) = \sum_{j=1}^{k} c_j = \sum_{c_j>0} (c_j - 1) + q, \tag{1}$$

if P is covered by rows in q directions. Define

$$\eta_j(P) = \begin{cases} c_j - 1, & c_j > 0, \\ 0, & c_j = 0, \end{cases}$$

for $j = 1,..., k$, and define

$$\epsilon_{ij}(P) = \begin{cases} 1 & \text{if } P \text{ is covered by rows in the directions } i \text{ and } j, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i < j \leqslant k$.

The q directions in which P is covered by rows include $\frac{1}{2}q(q - 1)$ pairs of directions. Hence

$$\sum_{i<j} \epsilon_{ij}(P) = \tfrac{1}{2}q(q - 1) \leqslant \tfrac{1}{2}k(q - 1), \tag{2}$$

and

$$q \geqslant 1 + \frac{2}{k} \sum_{i<j} \epsilon_{ij}(P).$$

Now (1) implies

$$m(P) \geqslant 1 + \sum_j \eta_j(P) + \frac{2}{k} \sum_{i<j} \epsilon_{ij}(P).$$

Let

$$\eta_j = \sum_P \eta_j(P), \qquad \epsilon_{ij} = \sum_P \epsilon_{ij}(P).$$

If S contains ν points, there are ν rook domains in the covering, each of which contributes nk to $\sum m(P)$. Hence, the sum of the above inequality over all $P \in V_n{}^k$ is

$$nk\nu \geqslant n^k + \sum_j \eta_j + \frac{2}{k} \sum_{i<j} \epsilon_{ij} . \tag{3}$$

Next we will show that

$$\epsilon_{ij} \geqslant \frac{\nu^2}{n^{k-2}} - \frac{\nu}{n^{k-1}} (\eta_i + \eta_j) + \frac{1}{4n^k} (\eta_i + \eta_j)^2 - \frac{\eta_i + \eta_j}{4} . \tag{4}$$

Consider the n^{k-2} planes in the (i, j)-th direction, numbered in some order.

Let the r-th plane contain α_r points of S, lying on a_r rows in direction i and on b_r rows in direction j. Then if

$$\eta_{ir} = \alpha_r - a_r\,, \qquad \eta_{jr} = \alpha_r - b_r\,,$$

we have

$$\eta_i = n \sum_r \eta_{ir}\,, \qquad \eta_{jr} = n \sum_r \eta_{jr}\,.$$

Since $0 \leqslant a_r\,, b_r \leqslant n$,

$$|\,\eta_{ir} - \eta_{jr}\,| \leqslant n.$$

By definition,

$$\epsilon_{ij} = \sum_r a_r b_r = \sum_r (\alpha_r - \eta_{ir})(\alpha_r - \eta_{jr})$$

$$= \sum_r \left[\left(\alpha_r - \frac{\eta_{ir} + \eta_{jr}}{2}\right)^2 - \left(\frac{\eta_{ir} - \eta_{jr}}{2}\right)^2\right].$$

Applying Schwarz's inequality to the first part of the sum,

$$\epsilon_{ij} \geqslant \frac{1}{n^{k-2}}\left[\sum_r \left(\alpha_r - \frac{\eta_{ir} + \eta_{jr}}{2}\right)\right]^2 - \frac{n}{2}\sum_r \frac{\eta_{ir} + \eta_{jr}}{2}$$

$$= \frac{1}{n^{k-2}}\left[\nu - \frac{\eta_i + \eta_j}{2n}\right]^2 - \frac{1}{4}(\eta_i + \eta_j),$$

verifying (4).

Summing (4) for all $i < j$,

$$\sum_{i<j} \epsilon_{ij} \geqslant \frac{1}{2} k(k-1)\frac{\nu^2}{n^{k-2}} - (k-1)\frac{\nu}{n^{k-1}}\sum_j \eta_j - \frac{k-1}{4}\sum_j \eta_j$$

$$+ \frac{1}{4n^k}\sum_{i<j}(\eta_i + \eta_j)^2.$$

Apply this inequality in (3). The result may be written as

$$\left[\frac{(k-1)\nu}{n^{k-2}} - n\right]\left[n^{k-1} - \nu + \frac{2}{kn}\sum \eta_j\right]$$

$$\geqslant \frac{k-3}{2k}\sum_j \eta_j + \frac{1}{4kn^k}\sum_{i<j}(\eta_i + \eta_j)^2. \tag{5}$$

The right side of this inequality is non-negative. If $\nu < n^{k-1}/(k-1)$, the left side is negative. Hence

$$\nu \geqslant n^{k-1}/(k-1).$$

Equality here implies equality in (5) (the right side must be zero) as well as in all preceding inequalities. From (5), $\eta_j = 0$, $j = 1,\ldots, k$. From (2), at every point $q = 1$ or k. This means that the hypotheses of Lemma 1 are satisfied. Hence $n/(k-1)$ is an integer. This completes the proof of the theorem.

III. Consequences

The function $w(k, n)$ is defined as the maximum number of disjoint 1-dense sets in $V_n{}^k$ (see [1]). A simple corollary of the above theorem is:

Corollary. $w(k, n) \leqslant n(k-1)$. *Equality cannot hold unless n is divisible by $k-1$.*

Proof: Suppose that a family of j disjoint 1-dense sets exists in $V_n{}^k$. Each set must contain at least $n^{k-1}/(k-1)$ points. Hence

$$j \cdot \frac{n^{k-1}}{k-1} \leqslant n^k,$$

$$j \leqslant n(k-1).$$

If $j = n(k-1)$, each of the 1-dense sets must contain only $n^{k-1}/(k-1)$ points. Hence, n is divisible by $k-1$. This completes the proof.

This corollary was stated (in a weaker form) as a conjecture in [1]. Using the stronger form, three entries in the table of bounds for $w(k, n)$ given there can be improved. We have $w(5, 6) \leqslant 23$, $w(5, 10) \leqslant 39$, $w(9, 10) \leqslant 79$.

We remark that another conjecture [2] which would imply the above corollary remains unsolved.

IV. $(k-2)$-Dimensional Coverings

The theorem proved here, by induction on k, is complicated by the fact that diverse types of sets can be obtained when a rook domain is intersected with a coordinate hyperplane. The statement which must be proved by induction is stronger than the desired result. It concerns a covering of $V_n{}^k$ by a collection of $(k-2)$-dimensional sets of the types which can occur in a hyperplane section of a $(k-1)$-dimensional rook domain in V_n^{k+1}.

Definition. An $R^j_{k,l}$ is the union of the set of all j-dimensional coordinate planes in $V_n{}^k$ which contain any fixed l-dimensional coordinate plane $(0 \leqslant l \leqslant j \leqslant k)$. (In particular, a j-dimensional rook domain is an $R^j_{k,0}$.)

THEOREM 2. *For $k \geqslant 2$, let $V_n{}^k$ be covered by a collection C of $(k-2)$-dimensional objects consisting of ν_l $R^{k-2}_{k,l}$'s for $0 \leqslant l \leqslant k-2$. Then*

$$\sum_{l=0}^{k-2} \frac{k-l}{k} \nu_l \geqslant \frac{n^2}{k-1}.$$

Equality can be attained if and only if $h = n/(k-1)$ is an integer, and there are $k-2$ mutually orthogonal Latin squares of order h. If equality holds, the covering consists only of $R^{k-2}_{k,0}$'s.

PROOF (by induction on k): If $k = 2$, this inequality reduces to $\nu_0 \geqslant n^2$, which is clearly true, since the covering objects are points. Hence we may assume that $k \geqslant 3$, and that the lemma is true for V^{k-1}_n (for all n).

Let $i_1, \ldots, i_l$ be any set of $\leqslant k-2$ distinct integers from 1 to k. Define $\mu_{i_1 \cdots i_l}$ to be the number of objects in the covering which are $R^{k-2}_{k,l}$'s, in which the common l-dimensional plane has every coordinate but the i_1-th,..., i_l-th fixed. Then

$$\nu_l = \sum_{(i_1, \ldots, i_l)} \mu_{i_1 \cdots i_l},$$

where the sum is over all possible sets of l distinct integers $i_1, \ldots, i_l$ from 1 to k.

For $j = 1, \ldots, k$, if none of the subscripts $i_1, \ldots, i_l$ equals j, $\mu_{i_1 \cdots i_l}$ may be decomposed into n quantities

$$\mu_{i_1 \cdots i_l} = \sum_{m=1}^{n} \mu_{i_1 \cdots i_l}(j, m),$$

where $\mu_{i_1 \cdots i_l}(j, m)$ is the number of $R^{k-2}_{k,l}$'s contributing to $\mu_{i_1 \cdots i_l}$ in which the j-th coordinate has the value m in the common l-dimensional plane. Define

$$\epsilon_j = \min_{1 \leqslant m \leqslant n} \sum_{l} \sum{}_j \, \mu_{i_1 \cdots i_l}(j, m), \tag{6}$$

where the subscript j on the second summation sign indicates that only subscripts $i_1, \ldots, i_l$ different from j are to be used.

Let j be fixed, and pick m so that

$$\epsilon_j = \sum_{l} \sum{}_j \, \mu_{i_1 \cdots i_l}(j, m). \tag{7}$$

Denote the hyperplane in which the j-th coordinate is m by P. Consider the way in which P is covered by the covering of $V_n{}^k$.

The $R_{k,l}^{k-2}$'s contributing to ϵ_j in (7) intersect P in $R_{k-1,l}^{k-2}$'s. These $R_{k-1,l}^{k-2}$'s are formed by taking the unions of certain $(k-2)$-dimensional hyperplanes in P, from a set of hyperplanes which includes at most ϵ_j in each direction. Permuting the hyperplanes of $V_n{}^k$, if necessary, we may assume that these hyperplanes in P do not cut the cube

$$1 \leqslant x_p \leqslant n - \epsilon_j, \qquad p = 1,..., k-1, \tag{8}$$

where $x_1, ..., x_{k-1}$ are coordinates in P. This cube V' is a $V_{n-\epsilon_j}^{k-1}$, to which the lemma will be applied.

The elements of C which are not counted in (7) cover V'. Let $\mu'_{i_1\cdots i_q(j)}$, where the subscript j may be present, or not, and $i_1, ..., i_q \neq j$, be the number of elements of C which are counted in $\mu_{i_1\cdots i_q(j)}$, but not in (7). These sets intersect P in $R_{k-1,q}^{k-3}$'s, but the intersections with V' may be of various types. In each of the $R_{k-1,q}^{k-3}$'s, there is a common q-dimensional plane, in which $k-1-q$ of the x_p's are fixed. Let $\mu'^{(r)}_{i_1\cdots i_q(j)}$ be the number for which r of the fixed x_p's take values $> n - \epsilon_j$. These intersect V' in $R_{k-1,q+r}^{k-3}$'s. Applying the induction hypothesis to V',

$$\sum_{q=0}^{k-3} \sum_{i_1,...,i_q \neq j} \sum_{r=0}^{k-3-q} \frac{k-1-q-r}{k-1} (\mu'^{(r)}_{i_1\cdots i_q} + \mu'^{(r)}_{i_1\cdots i_q j}) \geqslant \frac{(n-\epsilon_j)^2}{k-2}. \tag{9}$$

If $\mu^{(r)}_{i_1\cdots i_q(j)}$ is defined similarly, but without the omission of those sets which contribute to (7), this implies that

$$\sum_{q=0}^{k-3} \sum_{i_1,...,i_q \neq j} \sum_{r=0}^{k-3-q} \frac{k-1-q-r}{k-1} (\mu^{(r)}_{i_1\cdots i_q} + \mu^{(r)}_{i_1\cdots i_q j}) \geqslant \frac{(n-\epsilon_j)^2}{k-2}. \tag{10}$$

It follows from (6) that

$$\sum_{\substack{s=1 \\ (s \neq j)}}^{k} \sum_{m=n-\epsilon_j+1}^{n} \sum_{l} \sum_{s} \mu_{i_1\cdots i_l}(s, m) \geqslant \epsilon_j \sum_{\substack{s=1 \\ (s \neq j)}}^{k} \epsilon_s .$$

Each element of C which contributes to $\mu^{(r)}_{i_1\cdots i_q(j)}$ is counted r times on the left in this inequality. Hence we have

$$\sum_{q=0}^{k-2} \sum_{i_1,...,i_q \neq j} \sum_{r=0}^{k-1-q} r\mu^{(r)}_{i_1\cdots i_q} + \sum_{q=0}^{k-3} \sum_{i_1,...,i_q \neq j} \sum_{r=0}^{k-1-q} r\mu^{(r)}_{i_1\cdots i_q j} \geqslant \epsilon_j \sum_{\substack{s=1 \\ (s \neq j)}}^{k} \epsilon_s .$$

Divide by $k - 1$ and add to (10). The result is

$$\sum_{q=0}^{k-2} \sum_{i_1,\ldots,i_q \neq j} \sum_{r=0}^{k-3-q} \frac{k-1-q}{k-1} (\mu^{(r)}_{i_1\cdots i_q} + \mu^{(r)}_{i_1\cdots i_q j})$$

$$+ \sum_{q=0}^{k-2} \sum_{i_1,\ldots,i_q \neq j} \sum_{r=k-2-q}^{k-1-q} \frac{r}{k-1} \mu^{(r)}_{i_1\cdots i_q}$$

$$+ \sum_{q=0}^{k-3} \sum_{i_1,\ldots,i_q \neq j} \sum_{r=k-2-q}^{k-1-q} \frac{r}{k-1} \mu^{(r)}_{i_1\cdots i_q j}$$

$$\geqslant \frac{(n-\epsilon_j)^2}{k-2} + \frac{1}{k-1} \epsilon_j \sum_{\substack{s=1 \\ (s \neq j)}}^{k} \epsilon_s . \tag{11}$$

Increasing the values of some of the coefficients on the left, this becomes

$$\sum_{q=0}^{k-2} \sum_{i_1,\ldots,i_q \neq j} \sum_{r=0}^{k-1-q} \frac{k-1-q}{k-1} \mu^{(r)}_{i_1\cdots i_q} + \sum_{q=0}^{k-3} \sum_{i_1,\ldots,i_q \neq j} \sum_{r=0}^{k-1-q} \frac{k-1-q}{k-1} \mu^{(r)}_{i_1\cdots i_q j}$$
$$\geqslant \frac{(n-\epsilon_j)^2}{k-2} + \frac{1}{k-1} \epsilon_j \sum_{s \neq j} \epsilon_s . \tag{12}$$

We have

$$\sum_{r=0}^{k-1-q} \mu^{(r)}_{i_1\cdots i_q (j)} = \mu_{i_1\cdots i_q (j)} .$$

Hence

$$\sum_{q=0}^{k-2} \sum_{i_1,\ldots,i_q \neq j} \frac{k-1-q}{k-1} \mu_{i_1\cdots i_q} + \sum_{q=0}^{k-3} \sum_{i_1,\ldots,i_q \neq j} \frac{k-1-q}{k-1} \mu_{i_1\cdots i_q j}$$
$$\geqslant \frac{(n-\epsilon_j)^2}{k-2} + \frac{1}{k-1} \epsilon_j \sum_{s \neq j} \epsilon_s .$$

Consider the result of summing this inequality over all values of $j = 1,\ldots, k$. The quantity $\mu_{i_1\cdots i_l}$ occurs in $k - l$ of the first sums, with coefficient $(k - 1 - l)/(k - 1)$, and in l of the second sums, with coefficient $(k - l)/(k - 1)$. Hence its coefficient in the resulting inequality is

$$\frac{(k-l)(k-1-l)}{k-1} + \frac{l(k-l)}{k-1} = k - l.$$

Thus we have

$$\sum_{l=0}^{k-2} \sum_{i_1,\ldots,i_l} (k-l)\, \mu_{i_1\cdots i_l} \geqslant \sum_{j=1}^{k} \left[\frac{(n-\epsilon_j)^2}{k-2} - \frac{\epsilon_j{}^2}{k-1} \right] + \frac{1}{k-1} \left(\sum_{s=1}^{k} \epsilon_s \right)^2,$$

or

$$\sum_{l=0}^{k-2} (k-l)\nu_l \geqslant \sum_{j=1}^{k} \left[\frac{n^2 - 2n\epsilon_j}{k-2} + \frac{\epsilon_j^2}{(k-1)(k-2)}\right] + \frac{1}{k-1}\left(\sum_{s=1}^{k} \epsilon_s\right)^2. \quad (13)$$

The terms on the right in (13) which are of second degree in the ϵ_j's form a positive definite quadratic form. Hence, the value of the right side, for unrestricted real values of the ϵ_j's, is a minimum when all the partial derivatives are zero:

$$\frac{\epsilon_j}{(k-1)(k-2)} - \frac{n}{k-2} + \frac{1}{k-1}\sum_{s=1}^{k} \epsilon_s = 0, \qquad j = 1,..., k.$$

The solution of these equations is $\epsilon_j = n/(k-1)$, $j = 1,..., k$. Inserting these values in (13), we obtain

$$\sum_{l=0}^{k-2} (k-l)\nu_l \geqslant \frac{kn^2}{k-1}, \quad (14)$$

which was to be proved.

If equality holds here, we must have equality in (9)–(12). First suppose $k = 3$. Comparing (11) and (12), we see that $\mu_j^{(1)} = 0$, $\mu^{(1)} = 0$. However, $\mu_j^{(0)} + \mu^{(0)} > 0$, for this number counts all the sets of C which contribute to the covering of V'. It follows that the hyperplanes (8) determining V' are unique, for if not they could be chosen in a different way to make $\mu_j^{(1)} + \mu^{(1)} > 0$. Next, if $k \geqslant 4$, equality in (9), by the induction hypothesis, implies that V' is covered only by $R_{k-1,0}^{k-3}$'s. This again means that the hyperplanes in (8) are unique, for otherwise they could be rechosen to make at least one of the sets covering V' an $R_{k-1,1}^{k-3}$.

Thus, for $k \geqslant 3$, equality in (14) implies that each set which contributes to (7) contains a $(k-2)$-dimensional hyperplane in every direction in P:

$$\mu_{i_1 \cdots i_l}(j, m) = 0 \qquad \text{if} \quad l > 0. \quad (15)$$

This is true for each j, for m such that (7) is valid. For arbitrary m, we have

$$\sum_{l} \sum_{j} \mu_{i_1 \cdots i_l}(j, m) \geqslant \epsilon_j, \quad (16)$$

which has the value $n/(k-1)$ if equality holds in (14). If this is true, summing over m and j yields

$$\sum_{l} (k-l)\nu_l \geqslant n \sum_{j=1}^{k} \epsilon_j = \frac{kn^2}{k-1}.$$

Hence, equality in (14) implies equality in (16) for all j, m. For any set of subscripts $i_1, \ldots, i_l$ $(l > 0)$, pick $j \neq i_1, \ldots, i_l$. Then (15) is true for every m. Summing over m, we have $\mu_{i_1 \cdots i_l} = 0$.

Now we know that equality can hold in (14) only if C consists of $n^2/(k-1)(k-2)$-dimensional rook domains. $n/(k-1)$ of these domains have their centers in each hyperplane. Any two of the centers have distance at least $k-1$. For, if not, a hyperplane containing them may be taken to be the plane P above, and the definition of V' would not be unique.

Simultaneous equality in (11) and (12) implies $\mu^{(k-2)} = 0$. On the other hand, if $\mu^{(r)} \neq 0$ for $1 \leqslant r \leqslant k-3$, the covering sets of V' include an $R_{k-1,r}^{k-3}$, which is impossible. Hence, $\mu^{(r)} = 0$ for $1 \leqslant r \leqslant k-2$. It follows that when the centers of C are projected into P, each point lies in one of the two regions

(i) $1 \leqslant x_p \leqslant n - \epsilon_j, \qquad p = 1, \ldots, k-1,$
(ii) $n - \epsilon_j + 1 \leqslant x_p \leqslant n, \qquad p = 1, \ldots, k-1.$

This projection consists of $n^2/(k-1)$ distinct points at distances $\geqslant k-2$. Hence the $n^2/(k-1)^2$ centers in the hyperplanes

$$\frac{(k-2)n}{k-1} + 1 \leqslant x_1 \leqslant n$$

in V_n^k all project into (ii). It follows that every $(k-3)$-dimensional plane in (ii) contains exactly one of the projections; hence, the point set obtained in (ii) by the projection of the centers of C is distinguished by the fact that any two points in this set are two members of a sequence of points of the set with successive distances $k-2$.

By considering other hyperplanes of V_n^k parallel to P, it is easily seen that the set of projections of the centers can be decomposed into $k-1$ sets of this type, each the projections of the centers in $n/(k-1)$ hyperplanes parallel to P. Hence, by choosing the coordinates $x_1, \ldots, x_k$ in V_n^k appropriately, the set of centers of C lies in the union of the $k-1$ disjoint cubes

$$\frac{tn}{k-1} + 1 \leqslant x_p \leqslant \frac{(t+1)n}{k-1}, \qquad p = 1, \ldots, k, \tag{17}$$

for $t = 0, 1, \ldots, k-1$. Each cube contains $n^2/(k-1)^2$ points at distance $\geqslant k-1$. Hence in each cube, each $(k-2)$-dimensional plane contains one center.

Conversely, if a set of $(k-2)$-dimensional rook domains in V_n^k is distributed in this way, it covers V_n^k. For, take any point of V_n^k. Of its k coordinates, there must be two which satisfy one of the $k-1$ sets of inequalities (17). Suppose, for example, that $x_1, x_2 \leqslant n/(k-1)$. Then the $(k-2)$-dimensional plane through the point in which x_1 and x_2 are fixed intersects the first cube, hence contains the center of a rook domain. This plane is part of the rook domain, so the given point is covered.

Thus, equality is possible if and only if $n/(k-1) = h$ is an integer, and $V_h{}^k$ contains h^2 points at distance $\geqslant k-1$. This condition is known to be equivalent to the existence of $k-2$ pairwise mutually orthogonal Latin squares of order h [1]. The proof is complete.

COROLLARY 2. A $(k-2)$-dense set in $V_n{}^k$ has at least $n^2/(k-1)$ elements. This number is possible if and only if $n/(k-1)$ is an integer, and there are $k-2$ orthogonal Latin squares of order $n/(k-1)$.

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